Kirchhoff rods with nonhomogeneous cross section

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Abstract

The Kirchhoff's theory for thin, inextensible, elastic rods with nonhomogeneous cross section is studied. The Young's and shear moduli of the rod are considered to vary radially, and it is shown that an analytical solution for the constitutive relations can be obtained for circular cross section and constant Poisson's ratio. We comment on possible applications of our results.

Key words: elastic rod model, nonhomogeneous cross section, constitutive relations, Kirchhoff rod model

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Filamentous structures are of great interest in both academic and industrial fields. From biological fibers and biopolymers to different kinds of wires and cables, the study of statics and dynamics of these systems has brought significant contribution to both the understanding of the life and nature, and the development of new technological devices.

The Kirchhoff rod model has been considered a good framework to study both statics and dynamics of filaments in Biology (Schlick, 1995; Olson, 1996; Wolgemuth *et al.*,

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2000; Goriely and Tabor, 1998; McMillen and Goriely, 2002; Tobias *et al.*, 2000; Fonseca and de Aguiar, 2001) and in Engineering (Sun and Leonard, 1998; Gottlieb and Perkins, 1999). In most cases, the rod is considered to be completely homogeneous. But some nonhomogeneities along the rod have been analyzed in the literature. Tridimensional conformations of nonhomogeneous rods may present chaotic behavior (Mielke and Holmes, 1988; Davies and Moon, 1993) and deviations from the helix pattern were shown to occur in the case of a rod with periodic variation of its Young's modulus (da Fonseca *et al.*, 2003). In order to investigate the effects of stiffness nonhomogeneity, da Fonseca and de Aguiar (2003) developed a method for finding equilibrium solutions of the static Kirchhoff equations for rods subjected to given boundary conditions. A geometric study of rods with varying cross section radius was performed by da Fonseca and Malta (2003). In these examples, the rod nonhomogeneities are function only of the arc-length along the rod axis (within a section transversal to the axis the system is homogeneous).

Here we shall deal with nonhomogeneities that are function only of the distance to the rod axis. In this case, points lying at a given distance from the rod axis will have the same mechanical properties even if they belong to different cross sections. The elasticity problem is defined as the specification of the so-called *state of stress*. It requires the knowledge of the stress at every point of the body (Love, 1934). The work by Zhang and Hasebe (1999) and by Chen *et al.* (2000) are examples of application of Elasticity theory to nonhomogeneous cylindrical rods. Zhang and Hasebe (1999) have obtained an exact solution for the stress of a radially nonhomogeneous hollow circular cylinder with exponential radial variation of the Young's modulus and constant Poisson's ratio. Here, we address the problem of obtaining an analytical solution for the stress of a nonhomogeneous rod within the approximations of the Kirchhoff's theory for thin rods. We shall show that an analytical solution can

be obtained for a rod of circular cross section presenting any kind of radial variation of the Young's modulus, but constant Poisson's ratio. Since the Kirchhoff's rod theory is largely used in modeling long, thin and inextensible elastic rods, our solution can be of interest in a large range of applications.

Examples of real systems presenting nonhomogeneous cross section are coaxial cables (Tang *et al.*, 2001), coated optical fibers (Li *et al.*, 2002) and the double stranded DNA molecule (Calladine and Drew, 1999). The process of coating structures, where thin layers of a given material are deposited on a given body, is largely used in industries as, for example, the process called nitriding that improves the so-called *tribo-mechanical* properties of engineering components (Miola *et al.*, 1998). Coating processes are also used in basic research. For example, Salvadori *et al.* (2003a,b) measured the Young's modulus of gold thin films deposited in Atomic Force Microscopy (AFM) cantilevers. They found that the Young's modulus of the gold thin films is about 12% smaller than its bulk elastic modulus. This information is interesting for the analysis we shall present below.

The results of the Kirchhoff model were recently derived by Mora and Müller (2003) through the rigorous method of Γ-convergence. They showed that the nonlinear bending-torsion theory for inextensible rods arises as the Γ-limit of three-dimensional nonlinear theory of elasticity. Nevertheless, we shall follow the more simple derivation of the Kirchhoff equations due to Dill (1992) since our final solution can be directly obtained using this approach. Our aim is to obtain an analytical solution for the so-called *constitutive relations* that relate the components of the moment to the components of a vector representing the deformations of the rod.

Dill (1992) presented the derivation of the Kirchhoff equations from the classical conservation laws of linear and angular momentum for a three-dimensional body

with a surface area A enclosing a volume V:

$$\begin{aligned} &\int_A \mathbf{p}_n dS + \int_V \mathbf{f} dV = \int_V \rho \ddot{\mathbf{X}} dV \ , \\ &\int_A (\mathbf{X} \times \mathbf{p}_n) dS + \int_V (\mathbf{X} \times \mathbf{f}) dV = \int_V \rho (\ddot{\mathbf{X}} \times \mathbf{X}) dV \ , \end{aligned} \tag{1}$$

where $\mathbf{p}_n$ is the contact force per unit area exerted on the oriented surface element $d\mathbf{S} = \mathbf{n} dS$, ρ is the mass density and $\mathbf{X}$ is the position with respect to a fixed origin (a dot indicates time derivative). External forces per unit volume acting on the body are represented by $\mathbf{f}$. We shall drop the derivatives with respect to time since dynamical problems will not be considered here.

The axis of the rod is defined as a smooth curve $\mathbf{x}$, in the 3D space, parametrized by the arc-length s: $\mathbf{x} = \mathbf{x}(s)$. A *director basis* is defined at each point of the curve, with $\mathbf{d}_3$ chosen as the tangent vector, $\mathbf{d}_3 = \mathbf{x}'$ (the prime denotes differentiation with respect to s). The orthonormal vectors, $\mathbf{d}_1$ and $\mathbf{d}_2$, lie in the plane normal to $\mathbf{d}_3$. We choose these vectors in such a way that $\mathbf{d}_1$,$\mathbf{d}_2$,$\mathbf{d}_3$ form a right-handed orthonormal basis at each point of the rod axis. The space variation of the director basis along the curve $\mathbf{x}$ is controlled by the *twist equation*: $\mathbf{d}_i' = \mathbf{k} \times \mathbf{d}_i$. The components in the director basis, k_1 and k_2, of the *twist vector*, $\mathbf{k}$, are the components of the curvature of the rod, and k_3 is the rod twist density.

In the Kirchhoff's theory the rod is seen as an assembly of short segments. Each segment is loaded by contact forces from the adjacent segments. The equations (1) are applied to each segment in order to obtain a one dimensional set of differential equations for the static and dynamics of the rod. Each segment of infinitesimal thickness will be referred to as a "cross section". The theory assumes that the cross section radius, at all points, is much smaller than both the total length and the curvature of the rod. Since equation (1) involves integration over the volume V of the

body or over the area A enclosing the volume V of the body, the nonhomogeneities in the cross section do not influence the main Kirchhoff equations. Only the constitutive relations may depend on the nonhomogeneities in the cross sections. The general equation relating stress and strain for a homogeneous isotropic material is (Love, 1934; Dill, 1992)

$$\tilde{S} = 2\mu\tilde{E} + \lambda(\mathrm{Tr}\tilde{E})\tilde{\mathbf{1}} \ , \tag{2}$$

where the *tilde* ˜ is used to denote a tensor. $\tilde{S}$ is the stress tensor, $\tilde{E}$ is the strain tensor and $\tilde{\mathbf{1}}$ is the unitary matrix. μ is the shear modulus and λ is one of the elastic constants of Lamé (Love, 1934; Dill, 1992). The equation (2) is valid for relatively small stresses where the linear theory of elasticity holds true.

Our problem consists of obtaining the constitutive relations for a rod made of a *nonhomogeneous isotropic material*. In order to clarify the idea of *nonhomogeneity* and *isotropy* of the material, we must remember that even though the length and the radius of curvature of the rod have to be much bigger than the cross section radius (for the validity of the Kirchhoff model), the rod is a three-dimensional body formed by elements of volume that are considered to be a continuous isotropic medium. Nevertheless, some of the elastic properties of the rod can vary from one element of volume to another, as for example, in the radial direction. Elasticity theory for nonhomogeneous isotropic media has been studied since 1960s (see, for example (Rostovtsev, 1964; Plevako, 1971; Chen *et al.*, 2001)).

Due to the fact that only rods with circular cross sections lead to an analytical solution for the constitutive relations, so we shall consider only radial variations for the elastic properties of the rod. As in (Rostovtsev, 1964; Chen *et al.*, 2001), the equation (2) remains valid for nonhomogeneous μ and λ. Here, $\mu = \mu(r)$ and

$\lambda = \lambda(r)$, where r is the distance of the point to the axis of the rod.

The point where the axis intersects the plane of the cross section is the origin of a Cartesian basis lying in this plane. This Cartesian basis can be defined by the two vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ of the director basis. The Young's modulus E is connected to the shear modulus μ through the Poisson's ratio, ν: $\nu + 1 = E/2\mu$.

The components ϵ_{ij} $(i, j = 1, 2, 3)$ of the strain tensor, $\tilde{E}$ (which is symmetric), within the Kirchhoff's theory, is given by (Dill, 1992):

$$
\begin{aligned}
\epsilon_{\alpha\beta} &= \tfrac{1}{2}(\tfrac{\partial u_\alpha}{\partial X_\beta} + \tfrac{\partial u_\beta}{\partial X_\alpha}) \ , \\
\epsilon_{\alpha 3} &= \tfrac{1}{2}(\tfrac{\partial u_3}{\partial X_\alpha} + (k_3 - k_3^{(0)})X_\beta) \ , \\
\epsilon_{33} &= (k_1 - k_1^{(0)})X_2 - (k_2 - k_2^{(0)})X_1 \ ,
\end{aligned}
\tag{3}
$$

where greek labels equal $1, 2$, and X_1 and X_2 are the components of the position vector of a material point of the rod in the Cartesian basis $(\mathbf{d}_1 \ , \mathbf{d}_2)$. $k_i^{(0)}$ is the i-th component of the twist vector, in the director basis, for the undeformed rod, also known as *intrinsic curvature*. u_i, $i = 1, 2, 3$, are the components of the displacement of the material point. The solution for $u_i = u_i(X_1, X_2)$ constitutes the solution for the elasticity problem.

In order to find the solutions for u_i $(i = 1, 2, 3)$ we shall use the local balance of momentum for the components of the stress tensor (Dill, 1992):

$$
\sum_{\alpha=1}^{2} \frac{\partial S_{\alpha l}}{\partial X_\alpha} = 0 \ , \quad l = 1, 2, 3 \ , \tag{4}
$$

where S_{ij} is the ij-th component of the stress tensor, $\tilde{S}$. The boundary conditions are provided by the load conditions on the rod lateral surface. In the Kirchhoff's

theory, it is given by

$$\sum_{\alpha=1}^{2} n_\alpha S_{\alpha l} = 0 \ , \quad l = 1, 2, 3 \ , \tag{5}$$

where n_1 and n_2 are the components of the unit outward vector, normal to the boundary of the undeformed cross section. Therefore, the equation (4) must be solved for u_i subjected to the boundary conditions defined in (5).

The equation (4) can be separated and solved in two sets. The first one includes all terms with the index $l = 3$ (equation (4)):

$$\sum_{\alpha=1}^{2} \frac{\partial S_{\alpha 3}}{\partial X_\alpha} = 0 \ , \tag{6}$$

$$S_{\alpha 3} = 2\mu(r)\epsilon_{\alpha 3} \ , \tag{7}$$

$$\epsilon_{\alpha 3} = \frac{1}{2}(\frac{\partial u_3}{\partial X_\alpha} + (k_3 - k_3^{(0)})X_\beta) \ . \tag{8}$$

The boundary conditions for this set are:

$$\sum_{\alpha=1}^{2} n_\alpha S_{\alpha 3} = 0 \ . \tag{9}$$

The distance to the origin, r, is connected to X_1 and X_2 through the polar transformation $X_1 = r\cos\theta$, $X_2 = r\sin\theta$ ($r = \sqrt{X_1^2 + X_2^2}$). Substituting equations (7) and (8) in equation (6), and using the following relation

$$\frac{\partial r}{\partial X_\alpha} = \frac{X_\alpha}{r} \ , \tag{10}$$

equation (6) reads

$$\frac{\partial}{\partial X_1}(\mu(r)\frac{\partial u_3}{\partial X_1}) + \frac{\partial}{\partial X_2}(\mu(r)\frac{\partial u_3}{\partial X_2}) = 0 \ . \tag{11}$$

This equation describes the torsion of a rod subjected to the boundary condition given by equation (9). The known solution is (Love, 1934; Dill, 1992):

$$u_3 = (k_3 - k_3^{(0)})\varphi(X_1, X_2) \ , \tag{12}$$

where $\varphi(X_1, X_2)$ is known as *warping* (Dill, 1992) or *torsion* (Love, 1934) function, and must satisfy:

$$\mu(r)(\frac{\partial^2\varphi}{\partial X_1^2} + \frac{\partial^2\varphi}{\partial X_2^2}) + \frac{1}{r}\frac{d\mu}{dr}(X_1\frac{\partial\varphi}{\partial X_1} + X_2\frac{\partial\varphi}{\partial X_2}) = 0 \ . \tag{13}$$

The boundary condition (9) becomes:

$$n_1(\frac{\partial\varphi}{\partial X_1} - X_2) + n_2(\frac{\partial\varphi}{\partial X_2} + X_1) = 0 \ , \tag{14}$$

and must be satisfied for all X_1 and X_2 such that $\sqrt{X_1^2 + X_2^2} = h$, h being the cross section radius.

In the homogeneous case, the function φ depends only on the geometry of the cross section. In the nonhomogenoeus case, it also depends on how the shear modulus μ varies with r. Nevertheless, by inspection of the equation (14) we see that the solution for the homogeneous case with circular cross section ($\varphi(X_1, X_2) = 0$) also satisfies the equation (13) and the boundary condition (14). Therefore, we consider here only rods with circular cross section.

The second set consists of the remaining equations:

$$\sum_{\alpha=1}^{2}\frac{\partial S_{\alpha\beta}}{\partial X_\alpha} = 0 \ , \tag{15}$$

$$S_{\alpha\beta} = 2\mu(r)\epsilon_{\alpha\beta} + \lambda(r)(\sum_{m=1}^{3}\epsilon_{mm})\delta_{\alpha\beta}, \ , \tag{16}$$

$$\epsilon_{\alpha\beta} = \frac{1}{2}(\frac{\partial u_\alpha}{\partial X_\beta} + \frac{\partial u_\beta}{\partial X_\alpha}) \ , \tag{17}$$

where $\beta = 1, 2$ and $\delta_{\alpha\beta}$ is the Kronecker delta. The boundary conditions for this set of equations are:

$$\begin{aligned} &\textstyle\sum_{\alpha=1}^{2} n_\alpha S_{\alpha 1} = 0 \ , \\ &\textstyle\sum_{\alpha=1}^{2} n_\alpha S_{\alpha 2} = 0 \ , \end{aligned} \tag{18}$$

If the Poisson's ratio, ν, is a constant within the cross section, it is possible to show that the solution for u_1 and u_2 in eqs. (15-17) have the same form of the homogeneous case, even for $\mu = \mu(r)$ and $\lambda = \lambda(r)$, and the boundary conditions (5) are satisfied. Since the stresses considered here are such that the rod remains inextensible, the assumption of constant Poisson's ratio is reasonable in spite of having varying Young's and shear moduli. In this case, the ratio of Young's over shear moduli must satisfy $\frac{E(r)}{2\mu(r)} =$ Constant $= \nu + 1$.

The explicit solutions for u_1 and u_2 are

$$\begin{aligned} &u_1(X_1, X_2) = -\nu(k_1 - k_1^{(0)})X_1X_2 + \tfrac{\nu}{2}(k_2 - k_2^{(0)})(X_1^2 - X_2^2) \ , \\ &u_2(X_1, X_2) = \nu(k_2 - k_2^{(0)})X_1X_2 + \tfrac{\nu}{2}(k_1 - k_1^{(0)})(X_1^2 - X_2^2) \ . \end{aligned} \tag{19}$$

Now, we can calculate the constitutive relations in terms of the components of the twist vector. The definition of the total moment of the cross section, $\mathbf{M}$, is

$$\mathbf{M} = \int_S \mathbf{r} \times \mathbf{p}_S \, dS \ , \tag{20}$$

where S is the area of the cross section, $\mathbf{r}$ is the position vector in the plane of the cross section, given by

$$\mathbf{r} = X_1\mathbf{d}_1 + X_2\mathbf{d}_2 \ , \tag{21}$$

and $\mathbf{p}_S$ is the contact force per unit area in the cross section that, in terms of the stress tensor, is given by:

$$\mathbf{p}_S = \mathbf{d}_3.\tilde{S} = S_{31}\mathbf{d}_1 + S_{32}\mathbf{d}_2 + S_{33}\mathbf{d}_3 \ . \tag{22}$$

Using the equations (21) and (22), and the fact that $\mathbf{M} = M_1\mathbf{d}_1 + M_2\mathbf{d}_2 + M_3\mathbf{d}_3$, we obtain:

$$M_1 = \int_S X_2 S_{33} \, dS \ , \tag{23}$$

$$M_2 = \int_S -X_1 S_{33} \, dS \ , \tag{24}$$

$$M_3 = \int_S (X_1 S_{32} - X_1 S_{31}) \, dS \ . \tag{25}$$

Finally, using the solutions for u_i given by the equations (19), and by the solution $u_3 = 0$ (we are considering circular cross section so that $\varphi(X_1, X_2) = 0$) we can obtain the components of the strain tensor using the equations (8) and (17). Substituting the components of the strain tensor in the equations (7) and (16) we can obtain the expressions for the components S_{31}, S_{32} and S_{33} of the stress tensor:

$$S_{31} = -\mu(r)(k_3 - k_3^{(0)})X_2 \ , \tag{26}$$

$$S_{32} = \mu(r)(k_3 - k_3^{(0)})X_1 \ , \tag{27}$$

$$S_{33} = E(r)((k_1 - k_1^{(0)})X_2 - (k_2 - k_2^{(0)})X_1) \ . \tag{28}$$

The constitutive relations for the components of the total moment of the cross section can be written in a final form as:

$$M_1 = (k_1 - k_1^{(0)})\pi \int_0^h E(r) r^3 dr \ , \tag{29}$$

$$M_2 = (k_2 - k_2^{(0)})\pi \int_0^h E(r) r^3 dr \ , \tag{30}$$

$$M_3 = (k_3 - k_3^{(0)}) 2\pi \int_0^h \mu(r) r^3 dr \ , \tag{31}$$

where h is the cross section radius.

Note that there is no constraint on the variation of the Young's or shear modulus with r (provided that the Poisson's ratio is constant).

The constitutive relations given by equations (29-31) have the same form of the homogeneous case. It means that within the approximations of the Kirchhoff's theory, the static and dynamics of a rod are not affected by radial nonhomogeneities in its cross sections. Our calculations constitute a demonstration that a thin nonhomogeneous rod behaves like a homogeneous one if the deformations have high radius of curvature as compared to the cross section radius.

The equations (29-31) can be used to compare the rigidity of homogeneous and nonhomogeneous rods with the same cross section radius. By comparing the expressions for M_1, M_2 and M_3 of homogeneous and nonhomogeneous cases, we can derive expressions for an *effective* Young's and shear moduli, E_{ef} and μ_{ef}, in terms of $E(r)$ and $\mu(r)$, respectively. By *effective* we mean the values for Young's and shear moduli which result in the same values for M_i ($i = 1, 2, 3$), if the rod was homogeneous. They are given by:

$$E_{ef} = \frac{4}{h^4} \int_0^h E(r) r^3 dr \ , \tag{32}$$

$$\mu_{ef} = \frac{4}{h^4} \int_0^h \mu(r) r^3 dr \ . \tag{33}$$

Consider the simple example of a rod like a coaxial cable, with the Young's modulus given by

$$E(r) = \begin{cases} E_0 \ \text{ for } \ 0 < r < r_0 \ , \\ \\ E_1 \ \text{ for } \ r_0 \leq r \leq h \ , \end{cases} \tag{34}$$

so that E_0 and E_1 are the Young's moduli of the inner and outer parts of the cross section defined by the regions from the origin to r_0 and from r_0 to h, respectively. Substituting the equation (34) in the equation (32), we obtain the following expression for the effective Young's modulus:

$$E_{ef} = E_0 \left(\frac{r_0}{h}\right)^4 + E_1 \left(1 - \frac{r_0^4}{h^4}\right) \ . \tag{35}$$

This simple expression can be used in various experimental set ups to find out the Young's modulus of several systems. For example, consider the experiment of Salvadori *et al.* (2003a,b) where the Young's modulus of gold thin films deposited in Atomic Force Microscopy (AFM) cantilevers were measured. We cannot apply our method directly to their experiments because the cross sections of the cantilevers are not circular, but our model provides another form of measuring the Young's modulus of gold thin films. If a thin film of gold (or another material) is made to grow on the cylindrical surface of a homogeneous rod, which Young's modulus E_0 is known, by measuring the bending coefficient of the coated rod (which is related to E_{ef}), the film thickness, $h - r_0$, and the total radius of the cross section, h, we can use equation (35) to obtain the Young's modulus, E_1, of the thin film. The only requirement is that the dimensions of the rod must be in accordance with the approximations of the Kirchhoff model.

Another interesting application is the study of the rigidity of superficial layers of

nitride and carbonitride compounds within steel due to absortion of nitrogen by a process called nitriding (Miola *et al.*, 1998). Our method can be used in two applications, in this case.

In the first application, as for the case of rods coated by thin films, we can measure the Young's modulus of the substance of the layer of nitride and carbonitride compounds by producing rods of steel (which Young's modulus is known), submitting the rod to the nitriding process and measuring the bending coefficient of the nitrided rod and the thickness of the layer. As in the previous case, the equation (35) gives the Young's modulus of the layer.

The second application of our method to nitrided rods is an indirect measurement of the thickness of the layer of nitride and carbonitride compounds through the measurement of the Young's modulus of the nitrided rod. In this case, the Young's modulus of the steel, and of the substance that composes the layer, must be previously known. We can obtain r_0 through equation (35).

In the Introduction, we mentioned that the Kirchhoff rod model has been used to study the elastic behavior of long pieces of DNA (Schlick, 1995; Olson, 1996; Tobias *et al.*, 2000). Since it is known that DNA is not a polymer with uniform cross section, our calculations guarantee that those studies about elasticity of long DNA's are not inconsistent with the use of the Kirchhoff rod model. Nevertheless, our approach could be also used to obtain information about the stiffness of the parts of the DNA. It is known that the most rigid part of the double helix DNA is the phosphate backbone chain (Calladine and Drew, 1999). The bases are connected, in a base-pair, through hydrogen bonds that are weak connections. Therefore, a complete base-pair in a DNA molecule is a net situation approximately close to the cross section of a coaxial cable. The inner part of DNA is formed by the bases that

are connected by weak hydrogen bonds and the outer part is formed by the strong phosphate backbone chains. If, by means of molecular approaches, it is possible to estimate the stiffness of the inner (outer) part of the DNA molecule, then the stifness of the outer (inner) part could be obtained, and the results could be checked by experimentally measured stiffness of long DNAs (see, for example, (Smith *et al.*, 1996)).

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